TOPOLOGICAL SEMIGROUPS AND UNIVERSAL SPACES RELATED TO EXTENSION DIMENSION

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Abstract. It is proved that there is no structure of left (right) cancelative semigroup on $[L]$-dimensional universal space for the class of separable compact spaces of extensional dimension $\leq [L]$. Besides, we note that the homeomorphism group of $[L]$-dimensional space whose nonempty open sets are universal for the class of separable compact spaces of extensional dimension $\leq [L]$ is totally disconnected.

1. Preliminaries

Let L be a CW-complex and X a Tychonov space. The *Kuratowski notation* $X\tau L$ means that, for any continuous map $f\colon A \to L$ defined on a closed subset A of X, there exists an extension $\bar{f}\colon X \to L$ onto X. This notation allows us to define the preorder relation $\preceq$ onto the class of CW-complexes: $L \preceq L'$ iff, for every Tychonov space X, $X\tau L$ implies $X\tau L'$ [1].

The preorder relation $\preceq$ naturally generates the equivalence relation $\sim$: $L \sim L'$ iff $L \preceq L'$ and $L' \preceq L$. We denote by $[L]$ the equivalence class of L.

The following notion is introduced by A. Dranishnikov (see, [5] and [4]). The *extension dimension* of a Tychonov space X is less than or equal to $[L]$ (briefly, $\mathrm{ext}-\dim(X) \leq [L]$) if $X\tau L$.

We say that a Tychonov space Y is said to be a universal space for the class of compact metric spaces X with $\mathrm{ext}-\dim(X) \leq [L]$ if Y contains a topological copy of every compact metric space X with $\mathrm{ext}-\dim(X) \leq [L]$. See [1] and [2] for existence of universal spaces.

In what follows we will need the following statement which appears in [3] as Lemma 3.2.

Proposition 1.1. *Let* $i_0 = \min\{i : \pi_i(L) \neq 0\}$. *Then* $\mathrm{ext}-\dim(S^{i_0}) \leq [L]$.

2. Main theorem

Recall that a semigroup S (whose operation is denoted as multiplication) is called a *left cancelation semigroup* if $xy = xz$ implies $y = z$ for every $x, y, z \in S$.

Theorem 2.1. *Let L be a connected CW-complex and Let Y be a universal space for the class of compact metric spaces X with* $\mathrm{ext}-\dim(X) \leq [L]$. *If* $\mathrm{ext}-\dim(Y) = [L]$, *then there is no structure of left (right) cancelation semigroup on Y compatible with its topology.*

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Proof. Suppose the contrary and let Y be a left cancelation semigroup. Let $\alpha(\coprod_{j=1}^{\infty} S_j^{i_0})$ be the Alexandrov compactification of the countable topological sum of copies $S_j^{i_0}$ of the sphere S^{i_0}, where $i_0 = \min\{i : \pi_i(L) \neq 0\}$. By the countable sum theorem for extension dimension and Proposition 1.1, $\mathrm{ext}-\dim(\alpha(\coprod_{j=1}^{\infty} S_j^{i_0})) \leq [L]$ and, since Y is universal, Y contains a copy of $\alpha(\coprod_{j=1}^{\infty} S_j^{i_0})$. We will assume that $\alpha(\coprod_{j=1}^{\infty} S_j^{i_0}) \subset Y$. Besides, since $\mathrm{ext}-\dim(Y) \geq [S^1]$, we see that Y contains an arc J. Let a, b be endpoints of J. There exists j_0 such that $aS_{j_0}^{i_0} \cap bS_{j_0}^{i_0} = \emptyset$. By Proposition 1.1, there exists a map $f\colon aS_{j_0}^{i_0} \cup bS_{j_0}^{i_0} \to L$ such that $f|aS_{j_0}^{i_0}$ is a constant map and $f|bS_{j_0}^{i_0}$ is not null-homotopic. Extend map f to a map $\bar{f}\colon Y \to L$. Let $g\colon [0,1] \to J$ be a homeomorphism, then the map $F\colon S_{j_0}^{i_0} \times [0,1] \to L$, $\bar{f}(x,t) = g(t)x$, is a homotopy that contradicts to the fact that $f|bS_{j_0}^{i_0}$ is not null-homotopic. ☐

The homeomorphism group $\mathrm{Homeo}(X)$ of a space X is endowed with the compact-open topology.

Theorem 2.2. *Suppose* $\mathrm{ext}-\dim(X) = [L]$ *and every nonempty open subset of* X *is universal for the class of separable metric spaces* X *with* $\mathrm{ext}-\dim(X) \leq [L]$. *Then the homeomorphism group* $\mathrm{Homeo}(X)$ *is totally disconnected.*

Proof. Suppose the contrary. Let $h \in \mathrm{Homeo}(X)$, $h \neq \mathrm{id}_X$. There exists $x \in X$ such that $h(x) \neq x$ and, therefore, there exists a neighborhood U of x such that $h(U) \cap U = \emptyset$. Since U is universal for the class of separable metric spaces X with $\mathrm{ext}-\dim(X) \leq [L]$, there exists an embedding of S^{i_0} into U, where i_0 is as in Proposition 1.1. We may suppose that $S^{i_0} \subset U$. There exists a map $f\colon S^{i_0} \cup h(S^{i_0}) \to L$ such that the restriction $f|S^{i_0}$ is not null-homotopic while the restriction $f|h(S^{i_0})$ is null-homotopic. Since $\mathrm{ext}-\dim(X) \leq [L]$, there exists an extension $\bar{f}\colon X \to L$ of the map f. The set

$$W = \{g \in \mathrm{Homeo}(X) : \bar{f}|g(S^{i_0}) \text{ is not null-homotopic } \}$$

is an open and closed subset of $\mathrm{Homeo}(X)$. We see that W is a neighborhood of unity that does not contain h. ☐

3. Open problems

Note that the case $L = S^n$ corresponds to the case of covering dimension. In this case, the topology of homeomorphism groups of some universal spaces has been investigated by many authors (see the survey [6]).

In particular, it is known (see [8] and [7]) that the homeomorphism group of the n-dimensional Menger compactum M^n (note that M^n satisfies the conditions of Theorem 2.2 with $L = S^n$) is one-dimensional.

Let $[L] \geq [S^1]$ and X be as in Theorem 2.2. Is $\dim(\mathrm{Homeo}(X)) \geq 1$?

Another version: Is there X that satisfies the conditions of Theorem 2.2 and such that $\dim(\mathrm{Homeo}(X)) \geq 1$?

References

[1] A. Chigogidze, *Cohomological dimension of Tychonov spaces.* Topology Appl. 79 (1997), no. 3, 197–228.

[2] A. Chigogidze, V. Valov, *Universal metric spaces and extension dimension.* Geometric topology: Dubrovnik 1998. Topology Appl. 113 (2001), no. 1-3, 23–27.

[3] A. Chigogidze, M. Zarichnyi, *On absolute extensors modulo a complex. Topology Appl. 86 (1998), no. 2, 169–178.*

[4] A. Dranishnikov, J. Dydak, Extension dimension and extension types. Tr. Mat. Inst. Steklova 212 (1996), Otobrazh. i Razmer., 61–94; translation in Proc. Steklov Inst. Math. 1996, no. 1 (212), 55–88

[5] A. N. Dranishnikov, *The Eilenberg-Borsuk theorem for mappings in an arbitrary complex. (Russian)* Mat. Sb. 185 (1994), no. 4, 81–90.

[6] Chigogidze, Alex; Kawamura, Kazuhiro; Tymchatyn, E. D. Menger manifolds. Continua (Cincinnati, OH, 1994), 37–88, Lecture Notes in Pure and Appl. Math., 170, Dekker, New York, 1995.

[7] L. G. Oversteegen, E. D. Tymchatyn, *On the dimension of certain totally disconnected spaces.* Proc. Amer. Math. Soc. 122 (1994), no. 3, 885–891.

[8] B. L. Brechner, *On the dimensions of certain spaces of homeomorphisms.* Trans. Amer. Math. Soc. 121 (1966) 516–548.

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